

# Boundary-Layer Behavior on Continuous Solid Surfaces:

## I. Boundary-Layer Equations for Two-Dimensional and Axisymmetric Flow

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This study deals with boundary-layer flow on continuous solid surfaces. Flow of this type represents a new class of boundary-layer problems, with solutions substantially different from those for boundary-layer flow on surfaces of finite length. In this paper the boundary-layer behavior on continuous surfaces is examined, and the basic differential and integral momentum equations of boundary-layer theory are derived for such surfaces. In subsequent papers these equations will be solved for the boundary layer on a moving continuous flat surface and a moving continuous cylindrical surface, for both laminar and turbulent flow in the boundary layer.

This paper is the first one of a series dealing with a study of boundary-layer behavior on continuous solid surfaces. A polymer sheet or filament extruded continuously from a die, or a long thread traveling between a feed roll and a wind-up roll, is an example of a moving continuous surface.

Consider a long continuous sheet which issues from a slot, as shown in Figure 2a. The moving continuous sheet is taken up by a wind-up roll. The slot and the wind-up roll are a finite distance apart and constitute the boundaries of the system. The assumption is also made that a certain time has elapsed after the initiation of motion so that steady state conditions prevail. Any flow disturbance created by the roll is neglected. An observer fixed in space will note that the boundary layer on the sheet originates at the slot and grows in the direction of motion of the sheet. The boundary-layer behavior here appears to be different from what would be expected if the sheet is considered as a moving flat plate of finite length on which the boundary layer would grow in a direction opposite to the direction of motion of the plate, or away from the leading edge of the plate, as shown in Figure 2b. This difference in boundary-layer behavior on a moving continuous surface and on a surface of finite length raises the question whether the results of investigations of boundary-layer behavior on moving surfaces of finite length are applicable to moving continuous surfaces.

A photograph of the boundary layer

on a continuous cylindrical surface is shown in Figure 1. The equipment consisted of a rectangular Lucite acrylic resin tank filled with birefringent liquid, a water solution of milling yellow, through which were pulled two continuous threads. The threads issue from holes in a block which is completely immersed in the tank. In the photograph shown the left-hand thread has been set in motion from top to bottom, whereas the right-hand thread is held stationary. The photograph of the boundary layer on the moving thread was taken with white light through crossed polaroids after steady state flow conditions had been attained. In this system the dark background represents regions of stationary fluid, or regions of zero shear. Note that the boundary layer grows in the direction of motion of the thread.

In this paper the boundary-layer behavior on continuous solid surfaces is discussed in some detail and compared with that on moving surfaces of finite length. The results of this study show that boundary-layer theory and the basic differential boundary-layer equations are still applicable to the motion of continuous surfaces. However the boundary conditions for the two cases are no longer the same, implying that the solutions of the differential equations of motion will be significantly different for these two cases.

Boundary-layer theory is the oldest branch of modern fluid mechanics; yet to date no investigation of boundary-layer behavior on continuous surfaces has been reported.

### THEORETICAL DEVELOPMENT

#### Boundary-Layer Behavior on Continuous Surfaces

*Comparison with Flat Surfaces of Finite Length.* Consider steady, two-dimensional, incompressible flow around a continuous solid surface moving in a fluid medium at rest, as shown in Figure 2a. An observer fixed in space will note that the boundary layer on the solid surface, which originates at the slot, grows in the direction of motion of the surface. At the solid surface the fluid moves in the  $x$  direction with a velocity ( $u$  component) equal to the velocity of the solid surface, whereas at increasing distance from the surface the velocity of the fluid in the  $x$  direction approaches zero asymptotically. The fluid velocity in the  $y$  direction ( $v$  component) varies from zero at the solid surface to some finite value at the edge of the boundary layer.

Consider now two solid particles on the moving surface, one located near the slot at  $x_1$  and the other located farther out at  $x_2$ . During the interval of time elapsed since both particles emerged from the slot, these two particles have been subjected to drag for different periods of time. The particle located at  $x_2$  has been subjected to drag for a longer period of time than that located at  $x_1$ . This characteristic will cause the boundary layer to grow in the direction of motion as observed. The same is not true however in the case of the moving flat surface of finite length; here, during the same interval of time, any two solid particles of the surface are subjected to drag for the same period of time.

Thus the essential physical characteristic of the boundary layer on a con-

tinuous surface is that the origin and termination of the boundary layer around such a surface are not identified with any part of the solid surface but are determined by the boundaries of the system. By contrast the boundary-layer limits on a moving surface of finite length are definitely identified with specific parts of the surface: namely the leading and trailing edges of the surface.

In Figure 2c the boundary layer on a stationary flat surface with the fluid stream moving over it from left to right is shown, as would be seen by an observer located on a stationary frame of axes. This case is the exact analogue of the case shown in Figure 2b and points to the existence of still another new case of boundary-layer flow. This is identical with the case of the moving continuous solid surface and involves the interaction of a finite fluid medium flowing over an infinite stationary solid



Fig. 1. Boundary layer on a continuous cylindrical surface.

surface. This can be visualized by considering the motion of a mass of fluid contained in an enclosure over a stationary long solid surface, as shown in Figure 3. The boundary-layer behavior over the stationary solid surface, as seen by an observer, moving with the enclosure will be identical to that on a moving continuous surface. Thus the cases shown in Figures 2a and 3 are identical and correspond to the two classical cases shown in Figures 2b and c.

**Comparison With a Plane Free Fluid Jet.** The fluid behavior around the continuous flat surface bears some similarity to the fluid behavior in a fluid jet. As is well known the volume rate of flow in a fluid jet increases with increasing distance from the source because fluid particles are carried away with the jet owing to friction on its boundaries. Hence the boundary layer in a fluid jet increases in the same direction as the direction of motion of the jet stream, as is the case with the moving continuous flat surface (Figure 2a). However whereas in the free

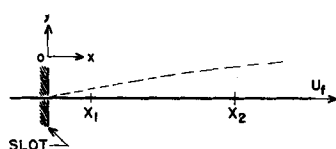
fluid jet the momentum flux is constant and independent of the distance from the origin, in the case of the moving continuous generated surface the momentum flux increases in the direction of motion. A continuous solid surface moving in a fluid medium may also be called a *solid jet*.

#### Boundary-Layer Equations of Motion for Laminar Flow

**Two-Dimensional Flow on a Continuous Flat Solid Surface.** The boundary-layer equations for laminar, steady, incompressible flow on a moving continuous flat solid surface or plane solid jet (Figure 2a) with no pressure gradient are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

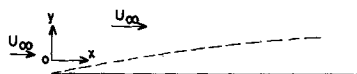
and



A. MOVING CONTINUOUS FLAT SURFACE



B. MOVING FLAT SURFACE OF FINITE LENGTH



C. STATIONARY FLAT SURFACE OF FINITE LENGTH

Fig. 2. Boundary-layer behavior on flat surfaces.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

The leading edge of the surface,  $x = 0$ , is located at the slot. The positive  $x$  axis is parallel to the solid surface and extends in the direction of motion of the surface. The  $y$  axis is positive upwards.

The boundary conditions are

$$u = U_1, v = 0 \text{ at } y = 0$$

$$u \rightarrow 0 \text{ at } y \rightarrow \infty$$

In addition  $(\partial u / \partial y)_{y=0} = 0$ .

The above equations are the same starting equations of motion as used for solid surfaces of finite length (2). However the boundary conditions for the flat plate of finite length are

$$u = 0, v = 0 \text{ at } y = 0$$

$$u \rightarrow U_\infty \text{ at } y \rightarrow \infty$$

Since the boundary conditions are different, the Blasius solution of Equations (1) and (2) for the flat plate of

finite length (2) will not be applicable to the case of the continuous flat surface. The solution of the equations of motion for the boundary layer on a moving continuous flat surface will be presented in Part II of this series.

**Axisymmetric Flow on a Continuous Cylindrical Solid Surface.** The boundary-layer equations for laminar, steady, incompressible flow on a moving continuous cylindrical solid surface or circular solid jet with no pressure gradient

are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (3)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \quad (4)$$

Equations (3) and (4) are to be solved subject to the boundary conditions

$$u = U_1, v = 0 \text{ at } r = a$$

$$u \rightarrow 0 \text{ at } r \rightarrow \infty$$

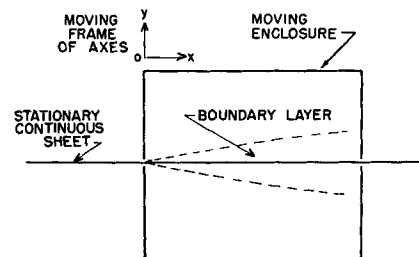


Fig. 3. Boundary-layer behavior on a stationary continuous flat surface.

In addition  $(\partial u / \partial r)_{r=\infty} = 0$ .

The solution of the equations of motion for the boundary layer on a moving continuous cylindrical surface will be presented in Part III of this series.

#### The Integral Momentum Equation for the Boundary Layer on a Continuous Surface

A complete calculation of the boundary layer for a given continuous solid surface by means of the differential equations of motion is, as will be shown in more detail in Parts II and III of this series, cumbersome and time consuming. One therefore desires an approximate method of solution to be applied in cases where an exact solution of the boundary-layer equations cannot be obtained with a reasonable amount of work, even if its accuracy is somewhat limited. This approximate method is devised so as to satisfy the exact boundary-layer equations in a region near the fluid-solid interface and near the region of transition to the external flow. In the remaining region of fluid in the boundary layer only a mean over the differential equation is satisfied, the mean being taken over the

whole thickness of the boundary layer. The resulting simplified approximate equation will henceforth be referred to as the *integral momentum equation of boundary layer for continuous surfaces*. This equation is similar to the momentum equation of boundary layer for solid surfaces of finite length, or what is otherwise known as von Karman's integral equation.

*The Integral Momentum Equation for Two-Dimensional Flow.* Consider steady, two-dimensional, incompressible flow around a continuous solid surface. Reference will be made to fluid motion adjacent to a moving continuous flat surface, which is described by Equations (1) and (2) with the appropriate boundary conditions.

Integrating Equation (1) with respect to  $y$  from  $y = 0$ , at the solid surface, to  $y = h$ , where the layer  $y = h$  is everywhere outside the boundary layer, one obtains

$$\int_{y=0}^{y=h} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = \left[ v \frac{\partial u}{\partial y} \right]_0^h \quad (5)$$

Substituting the shear stress at the solid surface for  $-(\mu/g_c)(\partial u/\partial y)_0$  in Equation (5), noting that  $v = \mu/\rho$ , one gets

$$\int_{y=0}^{y=h} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = \frac{g_c \tau_f}{\rho} \quad (6)$$

Replacing the normal velocity component from the continuity equation and integrating the left-hand side of the equation by parts, noting that

$$\left[ u \int_0^y \frac{\partial u}{\partial x} dy \right]_0^h = 0$$

one obtains

$$\int_{y=0}^{y=h} \left[ \frac{\partial}{\partial x} (u^2) \right] dy = \frac{g_c \tau_f}{\rho} \quad (7)$$

Since the integrand in Equation (7) vanishes outside the boundary layer, one can set  $h = \infty$ . Further differentiation with respect to  $x$  and integration with respect to  $y$  may be interchanged, as the upper limit  $h$  is independent of  $x$ . Defining

$$U_I^2 \theta = \int_{y=0}^{y=\infty} u^2 dy \quad (8)$$

and substituting in Equation (7) one gets the momentum equation for incompressible boundary layers on continuous flat surfaces:

$$\frac{d}{dx} (U_I^2 \theta) = \frac{g_c \tau_f}{\rho} \quad (9)$$

A quantity of interest is the displacement

\* The negative sign is used because the  $y$  coordinate is taken positive from the surface outwards, rather than from a space point towards the surface, as required in defining the stationary coordinate system.

ment thickness for the boundary layer on a continuous solid surface, which is defined by

$$\delta^* U_I = \int_{y=0}^{y=\infty} u dy \quad (10)$$

Since no statement was made concerning  $\tau_f$ , Equation (9) is applicable to both laminar and turbulent boundary layers.

Equation (8) is not identical with the integral momentum equation with no pressure gradient for the boundary layer on a flat solid surface of finite length (2). Hence known solutions of the momentum equation for surfaces of finite length would not be applicable to continuous surfaces.

*The Momentum Equation for Axisymmetric Flow.* The fluid motion around a continuous cylindrical surface is described by Equations (3) and (4) with the appropriate boundary conditions.

Integrating Equation (3) with respect to  $r$  from  $r = a$  at the solid surface to  $r = h$ , where the layer  $r = h$  is everywhere outside the boundary layer, one obtains

$$\int_{r=a}^{r=h} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} \right) r dr = \frac{a \tau_f}{(\rho/g_c)} \quad (11)$$

where the shear stress at the solid surface is  $\tau_f = -(\mu/g_c)(\partial u/\partial r)_{r=a}$ . The normal velocity component can be replaced by

$$v = -\frac{1}{r} \int_a^r \frac{\partial u}{\partial x} r dr$$

as seen from the continuity equation. Integrating the left-hand side of the equation by parts, noting that

$$\left[ u \int_a^r \frac{\partial u}{\partial x} r dr \right]_a^h = 0$$

one gets

$$\int_{r=a}^{r=h} \left[ \frac{\partial}{\partial x} (u^2) \right] r dr = \frac{a \tau_f}{(\rho/g_c)} \quad (12)$$

Since the integrand in Equation (12) vanishes outside the boundary layer, one can set  $h = \infty$ . When one defines a momentum area

$$\Theta = \frac{1}{U_I^2} \int_{r=a}^{r=\infty} u^2 2\pi r dr \quad (13)$$

and substitutes in Equation (12), the momentum equation for incompressible boundary layers on continuous cylindrical surfaces results:

$$\frac{d}{dx} (U_I^2 \Theta) = \frac{2\pi a \tau_f}{(\rho/g_c)} \quad (14)$$

The momentum thickness is related to the momentum area by

$$\Theta = \pi [(a + \delta)^2 - a^2] \quad (15)$$

Another quantity of interest is the

displacement area which is defined by

$$\Delta = \frac{1}{U_I} \int_{r=a}^{r=\infty} u 2\pi r dr \quad (16)$$

The displacement thickness is related to a displacement area by

$$\Delta = \pi [(a + \delta^*)^2 - a^2] \quad (17)$$

Equation (14) is applicable to both laminar and turbulent boundary layers. The momentum equation for the laminar boundary layer on a cylindrical solid surface of finite length has been considered by Glauert and Lighthill (1). The solutions developed by these investigators are not applicable however to continuous cylindrical solid surfaces.

## NOTATION

$a$	= radius of cylindrical surface, ft.
$g_c$	= conversion factor, 32.17 (lb.) (ft.)/(lb. force) (sec. <sup>2</sup> )
$h$	= distance from solid surface to a point outside the boundary layer in cartesian or cylindrical coordinates, ft.
$r$	= cylindrical coordinate from axis of cylindrical surface, ft.
$U_I$	= velocity of continuous solid surface, ft./sec.
$U_0$	= velocity of flat solid surface of finite length, ft./sec.
$U_\infty$	= velocity of potential flow over stationary flat solid surface of finite length, ft./sec.
$u$	= fluid-velocity component in $x$ direction, ft./sec.
$v$	= fluid-velocity component in $y$ or $r$ directions, ft./sec.
$x$	= cartesian or cylindrical axial coordinate, ft.
$y$	= cartesian coordinate from solid surface, ft.

## Greek Letters

$\Delta$	= displacement area, sq. ft.
$\delta$	= boundary-layer thickness, ft.
$\delta^*$	= displacement thickness, ft.
$\Theta$	= momentum area, sq. ft.
$\theta$	= momentum thickness, ft.
$\mu$	= fluid viscosity, (lb.)/(ft.) (sec.)
$\nu$	= fluid kinematic viscosity, sq. ft./sec.
$\rho$	= fluid density, lb./cu. ft.
$\tau_f$	= shear stress on continuous solid surface, lb. force/sq. ft.

## LITERATURE CITED

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Manuscript received January 2, 1959; revision received May 12, 1960; paper accepted May 16, 1960.